パーシステントホモロジーと表現論

Persistent homology and Representation theory

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@東北大学 MCCS交流会

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Topological Data Analysis

Topological Data Analysis ⋯ shape of Data

- ✱ Persistent Homology — Multiscale analysis
- ✱ Mapper — ≒ Reeb graph

main target

Remark Homology has been computable on computer but is Not robust for noises.

Homology counts the number of "holes" but do not understand how large they are.

⇝ Persistent Homology (about 2000 / Edelsbrunner, Letscher, Zomorodian (2002))

Preparation 1

$Q = (Q_0, Q_1, s, t)$: quiver = directed graph

vertex arrow source target

$A_n : 1 \xrightarrow{\alpha_{21}} 2 \longrightarrow \cdots \longrightarrow n$ type A

$Q_0 = \{1, 2, \cdots, n\}$, $Q_1 = \{\alpha_{i+1\, i} \mid i = 1, \cdots, n-1\}$, $s, t : Q_1 \longrightarrow Q_0$ s.t. $s(\alpha_{i+1\, i}) = i$, $t(\alpha_{i+1\, i}) = i+1$

☆ Q_0, Q_1 are not needed to be sets.

The composition $\beta \circ \alpha$ is defined if $t(\alpha) = s(\beta)$.

☆ Quiver + composition + identity = Category

☆ Quiver (finite) + α = f.d. algebra

Representations = modules

Representation theory of f.d. algebra

= To study the category of its modules.

Preparation 2

vect_k : the category of f.d. k-vector spaces & linear maps (k : field)

Representation over a quiver Q = Quiver morphism $Q \longrightarrow \mathrm{vect}_k$

category functor

Ex. A_n : $1 \xrightarrow{\alpha_{21}} 2 \longrightarrow \cdots \longrightarrow n$

Representation over A_n : $V_1 \xrightarrow{V_{21}} V_2 \longrightarrow \cdots \longrightarrow V_n$ V_i : vector space

$V_{i+1\, i}$: linear map

Interval representation $\mathbb{I}[b,d]$ (= indecomposable representation)

$$0 \to \cdots \to 0 \to \underset{\hat{b}}{k} \xrightarrow{\mathrm{id}_k} \cdots \longrightarrow \underset{\hat{d}}{k} \longrightarrow 0 \to \cdots \longrightarrow 0$$

Thm (Gabriel) representation over A_n $\overset{\exists!}{\cong} \bigoplus_{b \leq d} \mathbb{I}[b,d]$

Persistent Homology

Filtration of topological spaces $\mathbb{X}$: $X_1 \subset X_2 \subset \cdots \subset X_n$

⇩

Persistent Homology $H_q(\mathbb{X})$: $H_q(X_1) \longrightarrow H_q(X_2) \longrightarrow \cdots \longrightarrow H_q(X_n)$

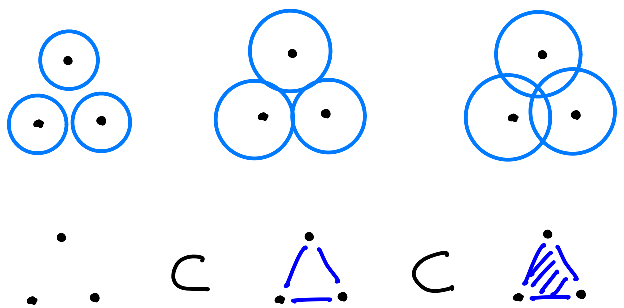
⇩

Representation over A_n

Persistence diagram : $PD(\mathbb{X}) = \{ (b,d) \mid H_q(\mathbb{X}) \simeq \bigoplus_{b \leq d} \mathbb{I}[b,d] \}$

<u>Ex.</u>

Point Cloud $\mapsto$ Čech Complex



ガラスについての研究例)

https://www.jst.go.jp/pr/announce/20160614/index.html

My interest

Multiparametric Persistent Homology

Aim : Consider Persistent Homology with 2 (or more) parameters

However, it is of infinite type (i.e, $\#\{\text{indec. rep.}\}/\cong \; = \infty$)

☆ Can not define a persistence diagram

⇝ We need a new invariant ! ⇨ Interval approximation

☆ Study a algebraic stability theorem (robustness for noises)

Thm $\quad d_B(PD(\mathbb{X}_1), PD(\mathbb{X}_2)) \leq d_I(H_q(\mathbb{X}_1), H_q(\mathbb{X}_2))$

⇝ We need a new viewpoint ! ⇨ Derived category

Remark

Ordinary (1-parameter)	Multiparameter
Module / $k[x]$ polynomial ring	Module / $k[x_1, x_2, \cdots, x_\ell]$
Representation / A_n quiver	Representation / $A_{n_1} \otimes A_{n_2} \otimes \cdots \otimes A_{n_\ell}$
Representation / $\mathbb{Z}$ or $\mathbb{R}$ poset	Representation / $\mathbb{Z}^\ell$ or $\mathbb{R}^\ell$
Constructible sheaf / $\mathbb{R}$	Constructible sheaf / $\mathbb{R}^\ell$

Thank you for your attention.