Harmonic Metrics and Zero Distribution of Holomorphic Sections

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Zeros of Holomorphic Functions

- Let $D \subseteq \mathbb{C}$ be a connected open subset, e.g.,

$$D = \{z \in \mathbb{C} \mid |z| < 1\},$$
$$D = \mathbb{C}^* = \mathbb{C} \setminus \{0\},$$
$$D = \mathbb{C}, \ldots$$

- A function $f : D \to \mathbb{C}$ is said to be *holomorphic* if f is complex differentiable at each point a of D. Equivalently, f is holomorphic if and only if f is smooth and satisfies the following Cauchy-Riemann equation on D:

$$\frac{\partial f}{\partial \bar{z}} = 0,$$
$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right).$$

- For example, a complex polynomial $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a holomorphic function over $\mathbb{C}$.
- Let f be a holomorphic function on D. A point $a \in D$ is said to be a *zero* of f if a satisfies $f(a) = 0$.
- The *zero set* of f is the set of all zeros of f.

The zeros of polynomials and holomorphic functions (or sections) have inspired significant mathematical study since ancient times:

- Orthogonal polynomials
- Dynamical systems on $\mathbb{CP}^1$
- Random polynomials (or sections)
- Holomorphic function spaces (Hardy space, Dirichlet space)
- Ramified coverings of Riemann surfaces
- Potential theory
- Fekete configurations, equilibrium measures
- Moduli spaces of n-points on $\mathbb{CP}^1$
- ...

Riemann Surfaces and Their Classification

A complex manifold X is said to be a *Riemann surface* if it has complex dimension one (real dimension two). Riemann surfaces are classified into the following three types from the viewpoint of curvature positivity:

1. Genus 0 Riemann surface: $\mathbb{CP}^1 = S^2$.
2. Parabolic Riemann surfaces: Elliptic curves (torus), $\mathbb{C}^*$, $\mathbb{C}$.
3. Hyperbolic Riemann surfaces: All others.

- A Riemann surface X is $\mathbb{CP}^1$ if and only if X admits a *positive* constant Gaussian curvature Kähler metric.
- A Riemann surface X is parabolic if and only if X admits a complete *flat* Gaussian curvature Kähler metric.
- A Riemann surface X is hyperbolic if and only if X admits a complete *negative* Gaussian curvature Kähler metric.

Harmonic Metrics

- *Harmonic metrics* on cyclic Higgs bundles are natural generalizations of flat Gaussian curvature Kähler metrics and constant negative Gaussian curvature Kähler metrics (although this perspective is not yet widespread).
- A rank r-*cyclic Higgs bundle* is defined in association with a holomorphic section $q \in H^0(K_X^r)$. For each cyclic Higgs bundle, we can associate the following elliptic equation, called the *Hitchin equation*:

$$\sqrt{-1}F_{h_j} + \mathrm{vol}(H_{j-1}) - \mathrm{vol}(H_j) = 0 \quad \text{for } j = 1, \ldots, r-1.$$

- From the solution to the Hitchin equation, we obtain r Kähler metrics $H_1, \ldots, H_r$.

- Suppose that q has no zeros. Then a flat Gaussian curvature Kähler metric provides a solution to the Hitchin equation.
- On the other hand, if q is identically zero, then a constant negative Gaussian curvature Kähler metric provides a solution to the Hitchin equation.
- I am examining the relationship between the zero configuration of a general q and the deviation of the Kähler metrics.

Ongoing Research

- I generalized the Hitchin equation and harmonic metrics to be associated with $\mathbb{Q}$-divisors and their limits (semipositive weights).
- I introduced the notion of *entropy*, which quantifies the deviation of the Kähler metrics $H_1, \ldots, H_r$ from flat Gaussian curvature Kähler metrics and constant negative Gaussian curvature Kähler metrics:

$$S(r, \varphi) := -\sum_{j=0}^{r-1} p_j(\varphi) \log p_j(\varphi),$$

$$p_j(\varphi) := \frac{\mathrm{vol}(H_j)}{\sum_{j=0}^{r-1} \mathrm{vol}(H_j)}.$$

- I am currently working on providing estimates for the entropy in various situations.