Estimation of Ocean Wave Spectra from Radar

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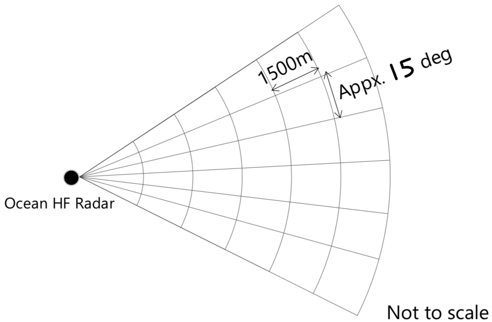
Motivation

- Understanding sea surface conditions is necessary for marine transportation.
- Observations of sea surface conditions can allow for safer boat navigation and warning of extreme weather events.
- In addition, sea surface conditions can help with identifying oil spills and debris.

Radar Range-Direction Grids

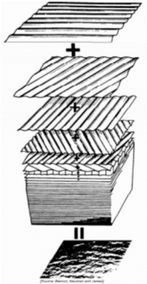


Radar Range-Direction Grids

- The width of the direction and range separations is determined by the radar's azimuth and range resolutions, which is approximately 15 deg and 1500m. The total azimuth is 120 deg and the total range is around 60 km.

Sea Surface Motion

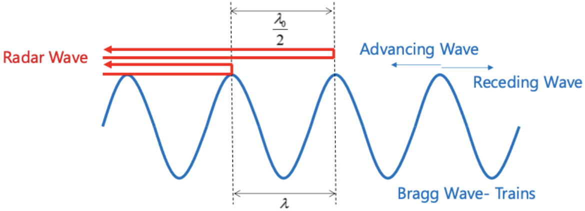
- Ocean waves on the sea's surface can be broken down in component waves, each with their own frequency and direction. These can represented by wavevectors.



Decomposition of Random Sea Surface Motions [1]

First Order Bragg Scattering

- When the radio wavelength is half of an ocean wavelength (aka the radio wave number is twice the ocean wave number), the constructive interference of the reflected radio waves is at its strongest and Bragg scattering occurs.



First Order Bragg Scattering

$$\sigma_{(1)}(\omega) = 2^6 \pi k_0^4 \sum_{m=\pm 1} S(-2m\mathbf{k}_0)\delta(\omega - m\omega_B) \tag{1}$$

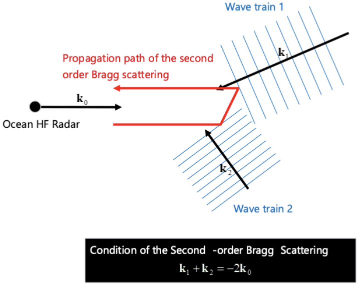
Second Order Bragg Scattering

- Second Order Bragg Scattering occurs due to electromagnetic and due to hydrodynamic effects. In the former, the radio wave is reflected twice, off of two different wave components. In the latter, the radio wave is reflected off the intersection of two ocean waves.
- Both types of second order Bragg scattering require that the two ocean wavevectors involved sum to twice the negative of the radio wavevector.

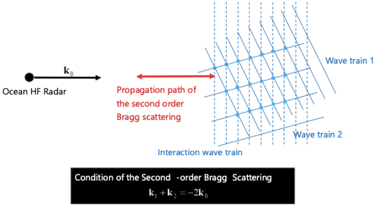
$$\mathbf{k_1} + \mathbf{k_2} = -2\mathbf{k_0} \tag{2}$$

- The Bragg scattered wave is receive by the radar as a Doppler spectrum. A Fourier transform is applied to convert it to a frequency domain equation ($\sigma_{(2)}(\omega)$ on slide 9).

Second Order Bragg Scattering



Electromagnetic effects



Hydrodynamic effects

Goal

$$\sigma_{(2)}(\omega) = 2^6 \pi k_0^4 \sum_{m_1=\pm 1} \sum_{m_2=\pm 1} \iint_{\mathbb{R}^2} |\Gamma(m_1\mathbf{k}_1, m_2\mathbf{k}_2)|^2 S(m_1\mathbf{k}_1) S(m_2\mathbf{k}_2) \times \delta(\omega - m_1\sqrt{gk_1} - m_2\sqrt{gk_2})\, dp\, dq \tag{3}$$

Aim

Given the Doppler spectrum for second-order Bragg scattering $\sigma_{(2)}(\omega)$, how can we determine the wave spectrum $S(\mathbf{k})$ using this theoretical model of the relationship between $\sigma_{(2)}(\omega)$ and $S(\mathbf{k})$.

Tikhonov Regularization

- We let $A(S)$ be the operator $L(\mathbb{R}^2, \mathbb{R}) \to L(\mathbb{R}, \mathbb{R})$ that takes a wave spectrum S as an input and outputs the resulting Doppler spectrum :

$$A(S) = 2^6 \pi k_0^4 \sum_{m_1=\pm 1} \sum_{m_2=\pm 1} \iint_{\mathbb{R}^2} |\Gamma(m_1 \mathbf{k}_1, m_2 \mathbf{k}_2)|^2 S(m_1 \mathbf{k}_1) S(m_2 \mathbf{k}_2) \times \delta(\omega - m_1 \sqrt{g k_1} - m_2 \sqrt{g k_2})\, dp\, dq \tag{4}$$

- Define cost function

$$L(S) = \|A(S) - \sigma_{(2)}(\omega)\|_{L^2(\mathbb{R})}^2 + \lambda \|S\|_{L^2(\mathbb{R}^2)}^2. \tag{5}$$

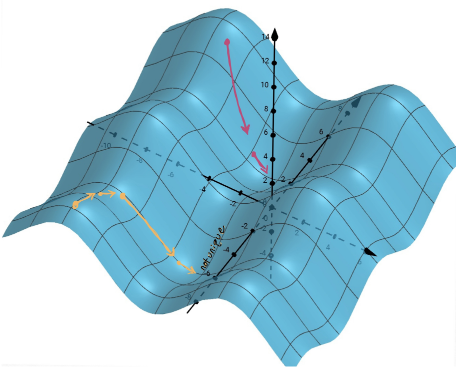
- We want to minimize this function over S.
- $\lambda > 0$ is a regularization parameter needed to avoid overfitting.

Optimize Using Gradient Descent

- Our strategy to minimize the cost function over S is to use gradient descent in order to attempt to find a minimizer of the cost function that closely approximates the correct wave spectrum.
- For the algorithm, we start with an initial guess S_0.
- We iterate by

$$S_{n+1} = S_n - t\nabla L(S_n) \qquad (6)$$

for a chosen step size t.



Necessary calculations for Gradient Descent

- Since we have an infinite dimensional input space, we need to define the gradient for our cost function.
- In order to do this, we need to use the Fréchet derivative:

$$\lim_{\|h\|_V \to 0} \frac{\|f(x+h) - f(x) - Df(h)\|_W}{\|h\|_V} = 0. \quad (7)$$

- The Riesz Representation Theorem says that for a linear operator $T : L^2(\mathbb{R}^2) \to \mathbb{R}$, there exists a $v \in L^2(\mathbb{R}^2)$ such that, for any h,

$$T(h) = \langle v, h \rangle_{L^2(\mathbb{R}^2)}. \quad (8)$$

- As the derivative is a linear operator, we can apply this to the Frechet derivative of L to get: $\exists v_{Df}$ such that for any h

$$Df(x)[h] = \langle v_{Df}(x), h \rangle. \quad (9)$$

- Since the directional derivative gives $D_u f(x) = \langle \nabla f(x), u \rangle$, we can conclude $\nabla f(x) = v_{Df}(x)$.

Calculation of Fréchet Derivative

- The Fréchet derivative of $L(S)$ is

$$L'(S)[h] = 2\langle A(S) - \sigma_{(2)}, A'(S)[h]\rangle_{L^2(\mathbb{R})} + 2\lambda\langle S, h\rangle_{L^2(\mathbb{R}^2)} \tag{10}$$

- As we can see, this requires us to calculate the derivative of $A(S)$, which we find to be:

$$\begin{aligned} A'(S)[h] =& 2^6\pi k_0^4 \sum_{m_1=\pm 1}\sum_{m_2=\pm 1}\iint_{\mathbb{R}^2} |\Gamma(m_1\mathbf{k}_1, m_2\mathbf{k}_2)^2| \Big[S(m_1\mathbf{k}_1)h(m_2\mathbf{k}_2) \\ &+ S(m_2\mathbf{k}_2)h(m_1\mathbf{k}_1)\Big]\delta(\omega - m_1\sqrt{gk_1} - m_2\sqrt{gk_2})\, dp\, dq. \end{aligned} \tag{11}$$

Calculation of Riesz Representation and Gradient

- The Riesz Representation Theorem gives that the Riesz representation of $L'(S)$ is a function $v_S \in L^2(\mathbb{R}^2)$ such that

$$L'(S)[h] = \iint_{\mathbb{R}^2} v_S(p,q)h(p,q)dpdq. \tag{12}$$

- In order to calculate v_s, we factor out h to get:

$$\nabla L(S) = \int_{-\infty}^{\infty} 4(A(S) - \sigma_{(2)}(\omega)| \times \sum_{m_1=\pm 1}\sum_{m_2=\pm 2} \left| \Gamma\left(-\frac{m_1}{m_2}(x+2m_2k_0),y),(x,y)\right)\right|^2 \times \delta\left(\omega - m_1\sqrt{g\left(\frac{1}{m_2}x+2k_0\right)^2+y^2} - m_2\sqrt{x^2+y^2}\right) \times S\left(-\frac{m_1}{m_2}(x+2m_2k_0,y\right)d\omega + 2\lambda S. \tag{13}$$

Applying Gradient Descent

- We start out with data consisting of discreet values from a Doppler spectrum of second order Bragg scattering $\sigma_{(2)}(\omega)$ where $\omega \in [-4.6116\pi, 4.6116\pi]$. To find S(**k**), we proceed with the following:

1. We interpolate these values in order to obtain a function on $\mathbb{R}$.
2. Chose an initial function S_0 (n=0).
3. Partition the *pq*-plane in order to numerically calculate (an approximation of) the integral of A(S_n)
4. Numerically calculate the Riesz representation.
5. Calculate the new value of S:

$$S_{n+1} = S_n - \alpha_n v_{S_n}. \tag{14}$$

6. Repeat steps 3-5 until the cost function is sufficiently small.

Tuning the Gradient Descent Algorithm

- In order to determine parameters for the gradient descent we will construct Doppler spectra from known wave spectra and try to reconstruct the starting wave spectra
- We want to use functions that could reasonably represent a physical wave spectra.
- We will consider a model that represents a dominant wavefront with a certain frequency and direction

What functions should we train the gradient descent on

- We use a model that separates the wave spectra into a frequency spectra and directional distribution [2]

$$S(f, \theta) = S_f(f) G(\theta | f) \tag{15}$$

- The frequency spectra is given by

$$S_f(f) = 0.257 H_s^2 T_s (T_s f)^{-5} e^{-1.03(T_s f)^{-4}} \tag{16}$$

where H_s is the significant wave height and T_s is the significant wave period

Directional Distribution

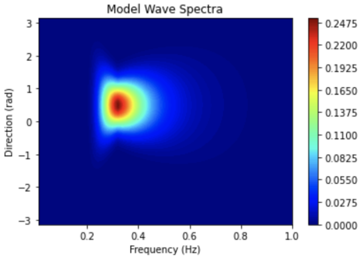
$$G(\theta|f) = G_0 \cos^{2\tilde{s}(f)}\left(\frac{\theta - \theta_0}{2}\right) \tag{17}$$

- θ_0 is the significant wave direction
- $\tilde{s}$ is the wave index given by

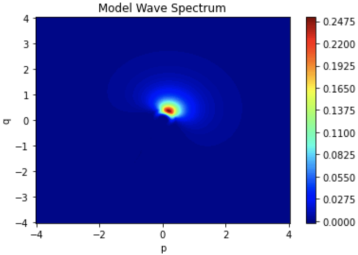
$$\tilde{s}(f) = \begin{cases} s_{max}\left(\frac{f}{f_p}\right)^5 & \text{if } f \leq f_p \\ s_{max}\left(\frac{f}{f_p}\right)^{-2.5} & \text{if } f > f_p \end{cases} \tag{18}$$

Example Spectrum

$$H_s = 1,\ T_s = 3,\ s_{\max} = 10,\ f_p = 0.317,\ \theta_0 = 0.5$$



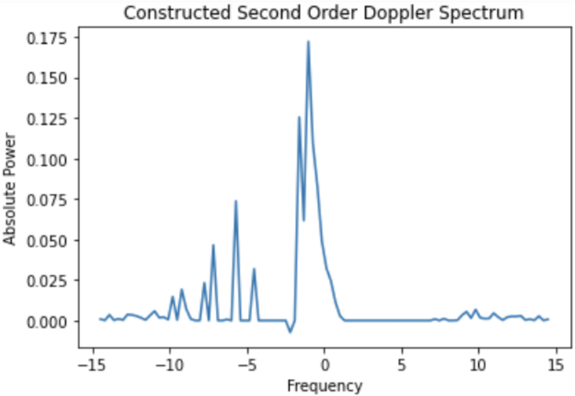
Model wave spectrum in frequency-direction plane



Model Wave spectrum in wave number plane

$$\theta = \arctan\left(\frac{p}{q}\right) \qquad f = \frac{1}{2\pi}\sqrt{g}(p^2 + q^2)^{1/4}. \tag{19}$$

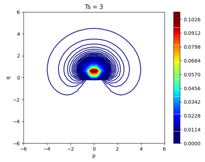
Constructed Second Order Doppler Spectrum



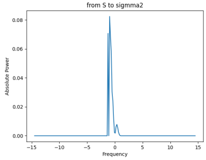
Constructed Second Order Doppler Spectrum

Numerical simulation for inverse problem

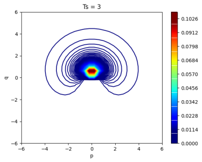
- Tried to get the initial condition from constructed $\sigma_{(2)}$ with $H_s = 1$, $T_s = 3$, $s_{\max} = 10$, $f_p = 0.317$, $\theta_0 = 0$ of this initial condition.
- These are the image of initial condition(max value is around 0.24), $\sigma_{(2)}$ constructed from initial condition and reconstruction from $\sigma_{(2)}$.



Initial wave spectrum.



Doppler spectrum constructed from initial wave spectrum.

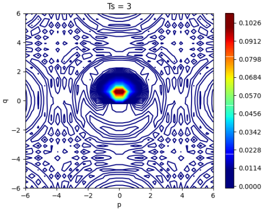


reconstruction

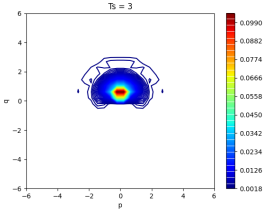
Parameters for reconstruction are step size $= 0.1$, iteration number $= 30$, regularization term $= 10^{-4}$ and the time for calculations is about 54 sec.

Numerical simulation for inverse problem

- Next we add perturbation to true data by sin curve and we get these reconstruction. We express the amplitude by ϵ.
- $\epsilon = 0.001$ and after changing the value less than 0.002 to 0. We calculated it by changing step number to 200 and the time for calculation is 267 sec.



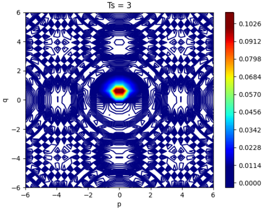
initial data with perturbation.



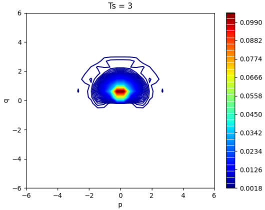
Reconstruction after setting values less than 0.002 to 0.

Numerical simulation for inverse problem

- $\epsilon = 0.003$ and after changing the value less than 0.0 to 0. We calculated it by same conditions of $\epsilon = 0.001$ and the time for calculation is 263 sec



initial data with perturbation.

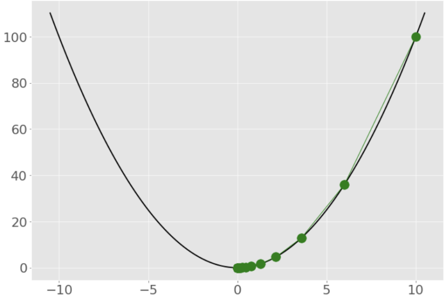


Reconstruction after setting values less than 0.0 to 0.

- In the future, we need to construct stronger system to reconstruct wave spectrum.

Dynamic Step Size

- In order to get the gradient descent method to converge faster rate of convergence for the gradient descent, we would like to use dynamic step size.
- We want large step sizes when we far from minima and small step sizes when we are close to minima



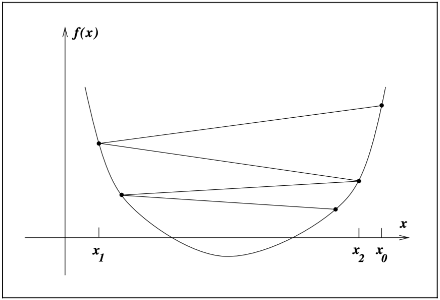
Example of gradient descent.

Decreasing Cost Function

- We want to ensure that the value of the cost function decreases with each iteration

$$L(S_{n+1}) < L(S_n) \tag{20}$$

- However, this condition is not sufficient to ensure quick convergence



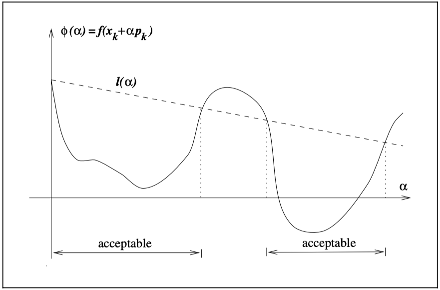
Gradient descent algorithm with insufficient decrease. Reprinted from [3].

Armijo Condition

- We introduce another term to demand further decrease

$$L(S_{n+1}) \leq L(S_n) - c_1\alpha_n\|\nabla L(S_n)\|_{L^2(\mathbb{R}^2)} \tag{21}$$

for $c_1 \in (0, 1)$ and α_n is the step size.

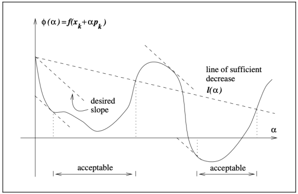


Example of Armijo Condition. Reprinted from [3].

- We also want the directional derivative to decrease in magnitude

$$\langle \nabla L(S_{n+1}), \nabla L(S_n) \rangle_{L^2(\mathbb{R}^2)} \geq -c_2 \|\nabla L(S_n)\|^2_{L^2(\mathbb{R}^2)} \quad (22)$$

for $c_2 \in (c_1, 1)$



Example of Wolfe Condition. Reprinted from [3].

- The strong Wolfe condition excludes slopes that are too positive as well

$$|\langle \nabla L(S_{n+1}), \nabla L(S_n) \rangle_{L^2(\mathbb{R}^2)}| \leq c_2 \|\nabla L(S_n)\|^2_{L^2(\mathbb{R}^2)} \quad (23)$$

Existence of Desired Step Size

- We assume that the cost function L is bounded below
- Define

$$l(\alpha) = L(S_n) - c_1\alpha\|\nabla L(S_n)\|_{L^2(\mathbb{R}^2)} \qquad \phi(\alpha) = L(S_n + \alpha\nabla L(S_n)) \tag{24}$$

- Note that $l'(0) > \phi'(0)$, so there exists an α_1 such that $l(\alpha_1) > \phi(\alpha_1)$.
- However, ϕ is bounded below, but l is not, so there exists an α_2 where $l(\alpha_2) < \phi(\alpha_2)$.
- Therefore, there exists a $\alpha \in (\alpha_1, \alpha_2)$ where $l(\alpha) = \phi(\alpha)$.
- Let α' be the smallest such α.
- For $\alpha \in (0, \alpha')$, the Armijo condition is satisfied.

Existence of Desired Step Size Continued

- For $\alpha \in (0, \alpha')$, the Armijo condition is satisfied.
- By the mean value theorem, there exists an $\alpha'' \in (0, \alpha')$ such that

$$\phi(\alpha') - \phi(0) = \alpha' \phi'(\alpha'') \tag{25}$$

$$l(\alpha') - L(S_n) = -\alpha' \langle \nabla L(S_{n+1}), \nabla L(S_n) \rangle \tag{26}$$

$$-c_1 \alpha' \|\nabla L(S_n)\|^2_{L^2(\mathbb{R}^2)} = -\alpha' \langle \nabla L(S_{n+1}), \nabla L(S_n) \rangle. \tag{27}$$

$$c_1 \|\nabla L(S_n)\|^2_{L^2(\mathbb{R}^2)} = |\langle \nabla L(S_{n+}), \nabla L(S_n) \rangle| \tag{28}$$

- $c_2 > c_1$, so

$$c_2 \|\nabla L(S_n)\|^2_{L^2(\mathbb{R}^2)} \geq |\langle \nabla L(S_{n+1}), \nabla L(S_n) \rangle|. \tag{29}$$

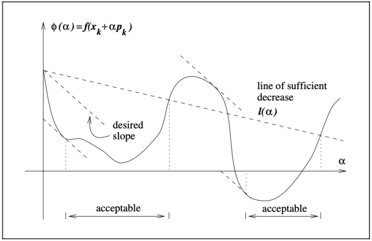
- Therefore, the Wolfe condition is satisfied for α''.

Backtracking Line Search

- For each iteration,

$$\alpha_n = M\beta^{m_n} \tag{30}$$

where M is the maximum possible step length, $\beta \in (0, 1)$, and m_n is the first natural number such that both the Armijo and Wolfe conditions are true.

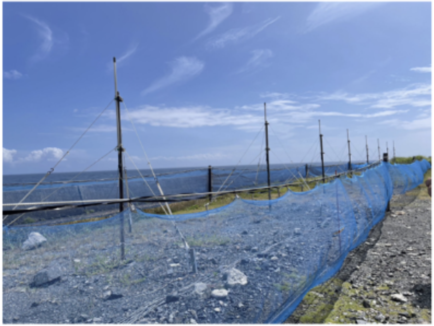


Example of Wolfe Condition. Reprinted from [3]

Environmental aspects of the study

- The main goal for studying the environmental effects is to determine a good initial guess for our gradient descent.

Throughout site visit...

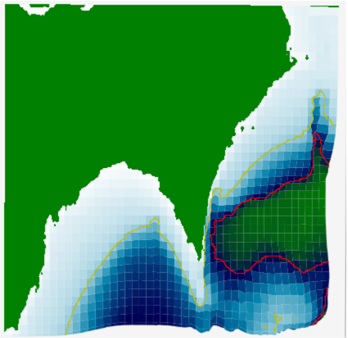


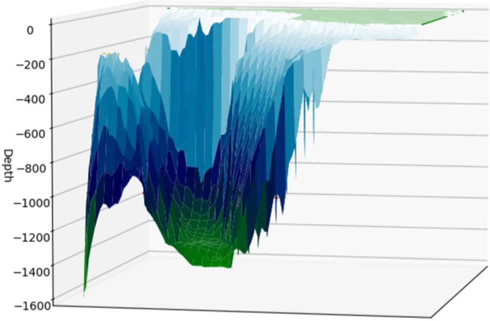
Send the wave spectra with four transmitting array antennas (left side) and receive the reflected wave spectra with eight receiving array antennas (right side). One radar is composed of these transmit and receive antennas. They are placed a few meters from the coast.

Throughout site visit...

- Saw the border between darker and lighter blue area of the ocean from airplane.
- Information about fishes and animals from deep sea around Kochi such as whales. Also the facility for pumping up deep sea water.
- Experienced strong wind from sea.
- Unique Terrain around the cost and the existence of the geopark.

Environmental aspects of the study



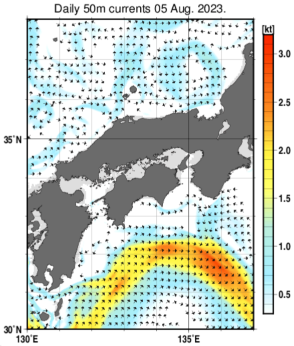


Topographic map of Shikoku region

Environmental aspects of the study

- Waves move faster in deeper water but as they enter shallow water, they slow and conservation of energy increases wave height.
- The slowing waves causes refraction which can change the direction of the wave fronts.

Environmental aspects of the study



Ocean current map. Reprinted from [4].

Waves tend to align with the currents. We will use this fact to determine the wave direction of our initial guess.

Summary

- We attempted to implement a gradient descent algorithm to solve the problem
- We hoped to optimize the gradient descent algorithm by adding dynamic step sizes and by using environmental factors to get a good initial guess
- Hopefully, these factors can be implemented in the future to obtain more efficient algorithm
- Then the knowledge of the wave spectra can be used to help with fishing and transportation

References

[1] J.M.J. Journée and W.W. Massie. *Offshore Hydromechanics.* Delft University of Technology, 2001.

[2] Tomoya Kataoka. "A Bayesian Inversion for a Directional Spectrum of Ocean Waves in Shallow Water Using HF Radar". In: (2016).

[3] Jorge Nocedal and Stephen J. Wright. *Numerical Optimization.* Springer, 2006.

[4] Japan Meteorological Agency. *Daily 50m Current.* 2023. URL: `https://www.data.jma.go.jp/kaiyou/data/db/kaikyo/daily/current_HQ.html?areano=8&stryy=2023&strmm=08&strdd=03`.

Thank you for listening